We review some facts without proof:

Theorem (Stirling approximation for $\Gamma(s)$)

Let $\delta > 0$. For $s \in \mathbb{C}$ with $|\arg s| \leq \pi - \delta$, we have

$$\log \Gamma(s) = \left(s - \tfrac{1}{2}\right)\log s - s + \frac{\log 2\pi}{2} + O\left(\frac{1}{|s|}\right)$$

and $\Gamma(s) = \sqrt{2\pi}\, s^{s-\frac{1}{2}} e^{-s} \left(1 + O\left(\frac{1}{|s|}\right)\right)$.

Corollary Fix σ_1, σ_2, t_0 real numbers with $\sigma_2 > \sigma_1$ and $t_0 > 0$. Then for $\sigma_1 \leq \sigma \leq \sigma_2$ and $|t| \geq t_0$,

$$|\Gamma(\sigma + it)| = \sqrt{2\pi}\, |t|^{\sigma - \frac{1}{2}} e^{-\frac{\pi}{2}|t|} \left(1 + O_{\sigma_1, \sigma_2, t_0}\left(\frac{1}{|t|}\right)\right).$$

Sketch of pf: For $\sigma_1 \leq \sigma \leq \sigma_2$ and $|t| \geq t_0$,

$$|\Gamma(\sigma+it)| \asymp_\sigma \frac{1}{|t|^{1/2}} \exp\left(\mathrm{Re}\left((\sigma+it)\left(\log\sqrt{\sigma^2+t^2} + i\arg(\sigma+it)\right)\right)\right)$$

$$\asymp_\sigma \frac{1}{|t|^{1/2}} e^{\sigma\left(\log|t| + O_\sigma\left(\frac{1}{|t|^2}\right)\right) - |t|\left(\frac{\pi}{2} + O_\sigma\left(\frac{1}{|t|}\right)\right)}$$

Application: From functional equation of $\zeta(s)$, we see that, for $\sigma > 0$,

$$\zeta(-\sigma + it) \ll_\sigma |t|^{\frac{1}{2}+\sigma}.$$

Definition: Let $f: \mathbb{C} \to \mathbb{C}$ an entire function. We say f has **bounded order** if $\exists M > 0$ such that

$$f(s) = O(\exp(|s|^M)), \text{ for all } s \in \mathbb{C}.$$

If f has bounded order, we denote the **order of f**

$$\inf\{\alpha \geq 0 : f(s) \ll \exp(|s|^\alpha)\}.$$

Notation: It is standard to denote the zeros of a complex function by ρ. By "$\sum_\rho$" or "$\prod_\rho$" we mean the sum / product over all zeros, unless specified differently.

Lemma: Let f an entire function of order α. Fix $\varepsilon > 0$. Then for all $R \geq 1$,

$$\sum_{|\rho| \leq R} 1 \ll R^{\alpha+\varepsilon}.$$

sum over zeros of f with $|\rho| \leq R$ counted with multiplicity.

Proof: Can assume without generality $f(s) \neq 0$, otherwise replace $f(s)$ with $f(s) s^{-m}$, for some m.

(from Jensen's ineq)

Then $$\sum_{|\rho| \leq R} 1 \leq \max_{|z| = 2R} \log \left| \frac{f(z)}{f(0)} \right| \ll (2R)^{\alpha+\varepsilon} \ll R^{\alpha+\varepsilon},$$

for all $\varepsilon > 0$. $\square$

Exercise: Show that if f entire of order α, then $\displaystyle\sum_{\rho:\,\rho\neq 0} |\rho|^{-\alpha+\varepsilon} < \infty$, for all $\varepsilon > 0$.

sum over non-zero zeros ρ.

Theorem (Hadamard / Weierstrass factorisation)

Let f be an entire function of bounded order α.

Then $$\boxed{f(z) = z^{\operatorname{ord}_{z=0} f}\, e^{g(z)} \prod_{\rho\neq 0}\left(1-\frac{z}{\rho}\right) e^{J\left(\frac{z}{\rho}\right)}},$$

where g is a polynomial of degree at most $\lfloor\alpha\rfloor$ and $\displaystyle J(x) = \sum_{j=1}^{\lfloor\alpha\rfloor} \frac{x^j}{j!}$.

(Think of it as a generalisation of Fundamental theorem of Algebra, we write an entire function as a product of its zeros.)

Example: $$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{l=1}^{\infty}\left(1+\frac{z}{l}\right) e^{-\frac{z}{l}}.$$

Notation: When we specialise to $\xi(s)$, we let "$\sum_{\rho}$" / "$\prod_{\rho}$" denote sum/product over non-trivial zeros!

<u>Theorem</u>: (Hadamard product for $\Xi(s)$).
Let $F(s) := s(s-1)\Xi(s)$.
Then $F(s) = e^{Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$,
for some constant B.

<u>Proof</u>: Recall $F(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$
is an entire function for $s \in \mathbb{C}$ and $F(s) = F(1-s)$.
Enough to estimate $F(s)$ for $\mathrm{Re}(s) \geq \frac{1}{2}$.

From Stirling approximation,

$$\log \Gamma\left(\frac{s}{2}\right) = \frac{s}{2} \log \Gamma\left(\frac{s}{2}\right) - \frac{s}{2} + \frac{1}{2}\log s + O(1)$$

(we are in $\mathrm{Re}(s) \geq \frac{1}{2}$, so $|\arg s| \leq \pi/2$).

Also, for $\mathrm{Re}(s) \geq \frac{1}{2}$, we have $(s-1)\zeta(s) \ll |s|^2$.

Therefore $F(s)$ is entire of order 1, hence it has Hadamard product. Its zeros are the non-trivial zeros of $\zeta(s)$.

$\Rightarrow$ Hadamard product $F(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$

Product over non-trivial zeros of $\zeta(s)$
counted with multiplicity

But note $e^{A} = F(0) = F(1) = \lim_{s \to 1} (s-1)\zeta(s) = 1$

$\Longrightarrow \underline{A = 0}$. $\square$

<u>Lemma</u>: $\sum_{\rho} \frac{1}{|\rho|^{\sigma}}$ converges for $\sigma > 1$, but diverges for $\sigma = 1$.

<u>Proof</u>: We know $\sum_{\rho} \frac{1}{|\rho|^{\sigma}}$ converges for $\sigma > 1$, since $F(s)$ is entire with order 1. (from exercise sheet).

Suppose for contradiction $\sum_{\rho} \frac{1}{|\rho|}$ converges.

We use the inequality $|(1-z)e^{z}| < e^{2|z|}$, for all $z \in \mathbb{C}$.

Then $F(s) = e^{Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} \ll e^{C|s|}$,

where $C = |B| + 2 \sum_{\rho} \frac{1}{|\rho|}$.

However, from Stirling approximation, for $\sigma > 0$ sufficiently large,

$$\Gamma\left(\frac{\sigma}{2}\right) = \sqrt{2\pi} \exp\left(\left(\frac{\sigma}{2} - \frac{1}{2}\right)\log\left(\frac{\sigma}{2}\right) - \frac{\sigma}{2}\right)\left(1 + O\left(\frac{1}{\sigma}\right)\right),$$

and hence $F(\sigma) > \exp\left(\frac{\sigma \log \sigma}{4}\right)$.

Contradiction. $\square$

Important remark: If ρ non-trivial zero of $\xi(s)$, so is $\bar{\rho}$, so we have $0 \leq \frac{1}{\rho} + \frac{1}{\bar{\rho}} = \frac{2\operatorname{Re}(\rho)}{|\rho|^2} \leq \frac{2}{|\rho|^2}$

So we have that $\lim_{x \to \infty} \sum_{|\rho| \leq x} \frac{1}{\rho}$ converges.

$\left(\sum_{|\rho| \leq x} \frac{1}{\rho} \leq \sum_{|\rho| \leq x} \frac{2}{|\rho|^2} \right.$, which converges as $\left. x \to \infty \right)$

We denote $\sum_{\rho} \frac{1}{\rho} := \lim_{x \to \infty} \sum_{|\rho| \leq x} \frac{1}{\rho}$.

! Order of summation important, not absolutely convergent !

Lemma: $B = - \sum_{\rho} \frac{1}{\rho}$.

Proof: Recall $F(s) = e^{Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$.

Taking logarithm derivative:

$$\frac{F'(s)}{F(s)} = B + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right).$$

Note that the poles of $\frac{F'}{F}$ are the zeros of $F(s)$, which are exactly the non-trivial zeros ρ of $\xi(s)$.

So $\sum_{\rho} \frac{1}{s-\rho}$ converges if s is not a non-trivial zero.

Since $F(s) = F(1-s)$, we have $\frac{F'(s)}{F(s)} = -\frac{F'(1-s)}{F(1-s)}$,

hence $\frac{F'(s)}{F(s)} = -B - \sum_{\rho}\left(\frac{1}{1-s-\rho} + \frac{1}{\rho}\right)$.

Therefore
$$B = -\sum_{\rho}\frac{1}{\rho} + \underbrace{\frac{1}{2}\sum_{\rho}\frac{1}{\rho-s} - \frac{1}{2}\sum_{\rho}\frac{1}{1-\rho-s}}_{=0}$$
$$= -\sum_{\rho}\frac{1}{\rho}.$$

We have used that if ρ is a zero of $\xi(s)$, so is $1-\rho$ (follows from functional equation).

$\square$

Lemma: (Partial fraction expansion for $\mathcal{L}(s)$).
Let $s = \sigma + it$, with $-1 \le \sigma \le 2$.
Then
$$\frac{\mathcal{L}'(s)}{\mathcal{L}(s)} = -\frac{1}{s-1} + \sum_{|\rho - s| \le 1}\frac{1}{s-\rho} + O(\log(2+|t|)).$$

Proof: Note that $\frac{\mathcal{L}'}{\mathcal{L}}(s)$ only has simple poles at the zeros of $\mathcal{L}(s)$ and at 1.

We first observe result follows for $|t| \le 10$, because $\frac{\mathcal{L}'}{\mathcal{L}}(s)$ is $O(1)$ unless close to one of the finite number of poles, in which case

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + O(1), \text{ if } s \text{ is close to } 1.$$

Note indeed that since $\zeta(s) = \frac{1}{s-1} + O(1)$, then

$\zeta'(s) = -\frac{1}{(s-1)^2} + O(1)$ (think Laurent expansion

of $\zeta'(s)$ around 1, and conclusion follows).

Similarly, $\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-\rho} + O(1)$ if s close to a zero ρ.

Now suppose $|t| \geq 10$. We have that

$$\frac{F'(s)}{F(s)} = \sum_{\rho} \frac{1}{s-\rho}.$$

But also $F(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$,

hence $\frac{F'(s)}{F(s)} = \frac{1}{s} + \frac{1}{s-1} - \frac{\log \pi}{2} + \frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \frac{\zeta'(s)}{\zeta(s)}$.

From Stirling formula, $\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O\left(\frac{1}{|s|}\right)$

when $|\arg(s)| \leq \pi - \delta$, true for us.

Therefore $\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho} \frac{1}{s-\rho} + O(\log|t|)$.

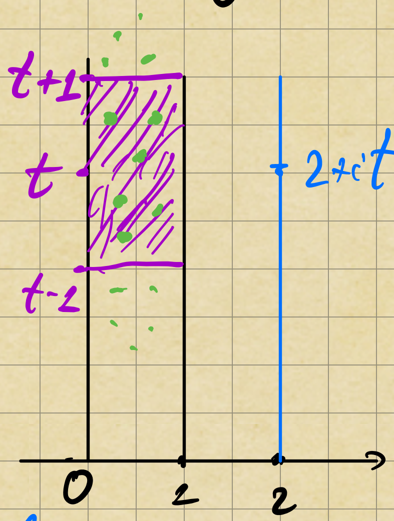
Recall $\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}$ for $\mathrm{Re}(s) > 1$, So $\frac{\mathcal{L}'(s)}{\mathcal{L}(s)} = O(1)$ for $\mathrm{Re}(s) \geq 2$.

Therefore

$$\frac{\mathcal{L}'(s)}{\mathcal{L}(s)} = \frac{\mathcal{L}'(s)}{\mathcal{L}(s)} - \frac{\mathcal{L}'(2+it)}{\mathcal{L}(2+it)} + O(1).$$

We rewrite this as $\frac{\mathcal{L}'(s)}{\mathcal{L}(s)} = \sum_{\rho} \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right) + O(\log|t|)$

We separate contributions of ρ's with $\mathrm{Im}(\rho)$ close to t or far from t.



When $|\mathrm{Im}\,\rho - t| \leq 1$, then

$$\sum_{\rho : |\mathrm{Im}\,\rho - t| \leq 1} \frac{1}{|2+it-\rho|} \leq \sum_{\rho : |\mathrm{Im}\,\rho - t| \leq 1} 1 \ll \log|t|$$

from estimation from last time of $N(T)$

When $|\mathrm{Im}\,\rho - it| > 1$, then $|s-\rho| \asymp |t - \mathrm{Im}\,\rho|$,

(since $-1 \leq \mathrm{Re}(s) \leq 2$)

so $\frac{1}{s-\rho} - \frac{1}{2+it-\rho} = \frac{2-\sigma}{(s-\rho)(2+it-\rho)} \ll \frac{1}{|t-\mathrm{Im}\,\rho|^2}$

Therefore the contribution from such ρ is

$$\sum_{\rho : |\mathrm{Im}\,\rho - t| > 1} \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right) \ll \sum_{\rho : |\mathrm{Im}\,\rho - t| > 1} \frac{1}{|t-\mathrm{Im}\,\rho|^2}$$

$$\ll \sum_{n\in\mathbb{N}} \sum_{n<|t-\operatorname{Im}\rho|<n+1} \frac{1}{|t-\operatorname{Im}\rho|^2} \ll \sum_{n} \frac{\log(|t|+n)}{n^2} \ll \log|t|.$$

Conclusion follows. $\square$

<u>Corollary</u>: (Size of $\frac{\zeta'}{\zeta}(s)$ controlled away from zeros)

Let $s=\sigma+it$ with $\sigma\geq -1$. If the distance from s to the nearest zero of $\zeta(s)$ is at least $\gg \frac{1}{\log(2+|t|)}$, then $\frac{\zeta'}{\zeta}(s) = O\left((\log(2+|t|))^2\right)$.

<u>Proof</u>: Follows easily from last lemma and that $|\{\rho : |\sigma+it-\rho|\leq 1\}| \ll \log(2+|t|)$.

$\square$